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Triality Principle for Semisimilarities

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Let M be a left vector space over the (commutative) field K and Q a nondegenerate quadratic form on M. A semilinear transformation S relative to the automorphism σ of K is called a σ-semisimilarity if, for any $x \in M$,

$$Q(xS) = \rho Q(x)^{\sigma} \tag{1}$$

where $\rho \neq 0$ is a fixed element of K called the ratio of S. Since Q is nondegenerate, S is invertible and S^{-1} is a semilinear transformation relative to σ^{-1}. From (1) we get

$$Q(xS^{-1}) = \rho^{-\sigma^{-1}} Q(x)^{\sigma^{-1}},$$

that is, S^{-1} is a σ^{-1}-semisimilarity of ratio $(\rho^{-1})^{\sigma^{-1}}$.

Now if C is a Cayley algebra over a field K of characteristic different from 2 corresponding to the nondegenerate quadratic form Q, there exists an involution $x \to \bar{x}$ in C such that $Q(x) = x\bar{x}$ (see [*3*], sect. 1). It is well known that given a similarity S of the underlying vector space of C relative to the quadratic form Q there exist similarities S_1 and S_2 such that either

$$(xy)S = (xS_1)(yS_2) \tag{2}$$

or

$$(xy)S = (yS_1)(xS_2) \tag{3}$$

holds for all $x, y \in C$.

We get (2) if and only if S is a proper similarity and (3) if and only if S is improper. In both cases S_1 and S_2 are uniquely defined up to a scalar factor and they are proper in case (2) and improper in case (3) (see [*2*], Thm. 1 and Cor. 1a, 1b).

Since S_1 and S_2 are defined up to a scalar factor, if $\bar{S}$ denotes the coset of S in the projective group of similarities $PS(Q)$ the mappings $\psi_i : \bar{S} \to \bar{S}_i$, $i = 1, 2$, are uniquely defined and their restrictions φ_i to the projective group of proper similitudes $PS^+(Q)$ define automorphisms of $PS^+(Q)$. If E is the improper involution of the orthogonal group $O(Q)$ defined by $xE = \bar{x}$, let ϵ denote the automorphism of $PS^+(Q)$, $\bar{U}^\epsilon = \overline{EUE}$. Then φ_1, φ_2, and ϵ generate a group isomorphic to the symmetric group S_3. (See [2], Cor. 2 of Thm. 1.)

In Theorem 1 of the present paper we extend formulas (2) and (3) to semisimilarities and we get as a consequence that the automorphisms φ_i, $i = 1, 2$, of $PS^+(Q)$ can be extended to automorphisms of the projective group of proper semisimilarities $P\Gamma S^+(Q)$. In Section 2 we apply this result to express the group of automorphisms of $PS^+(Q)$ as a semidirect product of $P\Gamma S^+(Q)$ by the symmetric group S_3.

1. We assume always that we are dealing with fields of characteristic $\neq 2$.

If M_1 and M_2 are subspaces of the vector space M with a nondegenerate quadratic form Q, a $1 - 1$ semilinear transformation U from M_1 onto M_2 relative to the automorphism σ is called a σ-semisimilarity of ratio ρ from M_1 to M_2 if

$$Q(yU) = \rho Q(y)^\sigma \quad \text{for all} \quad y \in M_1 .$$

We establish first an extension of Witt's theorem to semisimilarities (see, e.g., [1], Thm. 3.9).

Lemma 1. *Let M be a vector space with a nondegenerate quadratic form Q and let U be a σ-semisimilarity of ratio ρ from M_1 to M_2. Then U can be extended to a σ-semisimilarity of M is and only if there exist σ-semisimilarities of M of ratio ρ.*

Proof: The "only if" part is obvious, so we need prove that if there exists a σ-semisimilarity S of M of ratio ρ then U can be extended to M.

Let S^{-1} be the inverse of S. Then, for any $x \in M_1$, the mapping $x \to xUS^{-1}$ defines an isometry between M_1 and M_2S^{-1} for

$$Q(xUS^{-1}) = (\rho^{-1})^{\sigma^{-1}} Q(xU)^{\sigma^{-1}} = (\rho^{-1})^{\sigma^{-1}} (\rho Q(x)^\sigma)^{\sigma^{-1}} = Q(x).$$

By Witt's theorem, this isometry can be extended to an isometry T of M. Then for $x \in M_1$ $xT = xUS^{-1}$, that is, $xTS = xU$ and since TS is defined over M it gives an extension of U to M.

From now on Q will be the quadratic form $Q(x) = x\bar{x}$ of a Cayley algebra C. If a semilinear transformation relative to σ of the underlying vector space of C has the property that $(xy)S = (xS)(yS)$ we say that S is a σ-semiautomorphism of the algebra C.

Lemma 2. *If there exists a σ-semisimilarity S of ratio ρ of Q, then there exists a σ-semiautomorphism T of C.*

Proof: Let 1 be the identity element of C and $1S = x$. Then $Q(x) = Q(1S) = \rho Q(1)^\sigma = \rho$. If $R_{x^{-1}}$ is the similarity $yR_{x^{-1}} = yx^{-1}$ the transformation $S' = SR_{x^{-1}}$ is a σ-semisimilarity of ratio 1 which leaves invariant the element 1. Let e_1, e_2 be any two nonisotropic mutually orthogonal elements of the subspace orthogonal to 1; then the vectors $f_1 = e_1S'$ and $f_2 = e_2S'$ are mutually orthogonal and $Q(f_i) = Q(e_i)^\sigma \neq 0$, moreover $Q(f_1f_2) = Q(f_1)Q(f_2) = Q(e_1)^\sigma Q(e_2)^\sigma = Q(e_1e_2)^\sigma$. Hence the semilinear transformation T relative to σ defined by $1T = 1$, $e_1T = f_1$, $e_2T = f_2$, $(e_1e_2)T = f_1f_2$ is a σ-semisimilarity of ratio 1 from the underlying vector space of the quaternion subalgebra $M_1 = [1, e_1, e_2, e_1e_2]$ to the underlying vector space of the quaternion subalgebra $M_2 = [1, f_1, f_2, f_1f_2]$, even more, T is a "σ-semi-isomorphism" from the quaternion subalgebra M_1 to the quaternion subalgebra M_2. Now let e_3 be a nonisotropic vector in $M_1^\perp$, that is, e_3 belongs to the subspace orthogonal to M_1, then by Lemma 1 we know that there exists an element $f_3 \in M_2^\perp$ such that $Q(f_3) = Q(e_3)^\sigma$. If we extend T to a σ-semilinear transformation of the underlying vector space of C by defining

$$e_3T = f_3, \quad (e_1e_3)T = f_1f_3, \quad (e_2e_3)T = f_2f_3, \quad ((e_1e_2)e_3)T = (f_1f_2)f_3,$$

T is a σ-semiautomorphism of C.

THEOREM 1 (*Triality principle*). *If S is a σ-semisimilarity of Q of ratio ρ, then there exist σ-semisimilarities S_1 and S_2 such that either*

$$(xy)S = (xS_1)(yS_2) \qquad \text{or} \qquad (xy)S = (yS_1)(xS_2).$$

Moreover S_1 and S_2 are defined up to a scalar factor and only one of the above equalities holds.

Proof: By Lemma 2 we know that there exists a σ-semiautomorphism T of C, thus T is a σ-semisimilarity of ratio 1. Hence ST^{-1} is a similarity and since the triality principle holds for similarities we can find similarities U_1 and U_2 such that

$$(xy)ST^{-1} = (xU_1)(yU_2) \quad \text{if} \quad ST^{-1} \text{ is proper} \tag{4}$$

or

$$(xy)ST^{-1} = (yU_1)(xU_2) \quad \text{if} \quad ST^{-1} \text{ is improper.} \tag{5}$$

But T is a σ-semiautomorphism of C, therefore $(xy)S = (xU_1T)(yU_2T)$ if (4) holds and $(xy)S = (yU_1T)(xU_2T)$ if (5) holds. Taking $U_1T = S_1$ and $U_2T = S_2$ we get the first part of the theorem.

As to the uniqueness up to a scalar factor, this is an immediate consequence of the uniqueness of the decomposition for similarities.

We can define now the proper semisimilarities as those for which

$(xy)S = (xS_1)(yS_2)$ holds. Then it is clear that the proper semisimilarities form a subgroup of index 2 of the group of semisimilarities.

Remark. Given a quadratic form Q over an even dimensional space whose discriminant is a square (different from zero), if $x_1, x_2, \cdots, x_{2m}$ is an orthogonal basis such that $Q(x_1)Q(x_2) \cdots Q(x_{2m}) = 1$ we can define the proper (improper) σ-semisimilarities S as those for which the element $x_1x_2 \cdots x_{2m}$ of the Clifford algebra $C(Q)$ is mapped into itself (minus itself) under the σ-semiautomorphism of $C(Q)$ associated to S (see [*4*], section II). It is easily seen that this definition coincides with the preceding one because it can be proved that any σ-semiautomorphism of the Cayley algebra C is a proper σ-semisimilarity in the sense just defined. The proof is immediate when C is a split Cayley algebra and can be reduced to this case by extension of the base field if C is a division algebra.

Actually even when the discriminant of the nondegenerate quadratic form Q is not a square the distinction between proper and improper semisimilarities can be applied to the σ-semisimilarities if there exists an element β in the quadratic class of the discriminant of Q such that $\beta^\sigma = \beta$.

2. Let $\bar{S}$ denote the coset of the proper semisimilarity S in the projective group of proper semisimilarities $P\Gamma S^+(Q)$. By using the triality principle in the extended form given in Theorem 1, we can prove like in the case of similarities that the mappings $\bar{S}^{\varphi_i} = \bar{S}_i$, $i = 1, 2$, define automorphisms of $P\Gamma S^+(Q)$. Again, like in the case of similarities, if we substitute xy for x and $\bar{y}$ for y in $(xy)S = (xS_1)(yS_2)$ we get $Q(y)^\sigma(xS) = ((xy)S_1)(\bar{y}S_2)$ and multiplying on the right by $\overline{\bar{y}S_2} = yES_2E$ we obtain

$$(xy)S_1\rho = (xS)(\overline{\bar{y}S_2}) \quad \text{for any} \quad Q(y) \neq 0, \tag{6}$$

where $\rho = Q(yS_2)Q(y)^{-1}$ is the ratio of S_2. This implies that (6) is true for all $x, y \in C$.

Equality (6) says that $\bar{S}^{\varphi_1^2} = \bar{S}$ and $\bar{S}^{\varphi_1\varphi_2} = E\bar{S}^{\varphi_2}E = \bar{S}^{\varphi_2\epsilon}$. Similarly, substitution of $\bar{x}$ for x and xy for y gives $\bar{S}^{\varphi_2^2} = \bar{S}$ and $\bar{S}^{\varphi_2\varphi_1} = \bar{S}^{\varphi_1\epsilon}$. Thus the automorphisms φ_1, φ_2, and ϵ satisfy the relations $\varphi_1^2 = \varphi_2^2 = \epsilon^2 = 1$ and $\varphi_1\varphi_2 = \varphi_2\epsilon$, $\varphi_2\varphi_1 = \varphi_1\epsilon$, which imply that φ_1 and ϵ generate a subgroup isomorphic to the symmetric group S_3 (cf. [*2*], Cor. 2 of Thm. 1, notice that our notation is slightly different from the notation in that paper).

There is a simple interpretation of the isomorphism between S_3 and the group generated by ϵ and φ_1.

Let $\{D_0\}$ be the set of (6,2) cosets in $PS^+(Q)$, that is, cosets $\bar{U} \in PS^+(Q)$ which contain an orthogonal involution whose minus space has dimension 2. If U is the (6,2) involution in $\bar{U}$, let $a, b \in C$ be an orthogonal basis for its minus space, then

$$xU = Q(ab)^{-1}(b(\bar{a}x\bar{a})b)$$

and $\bar{U}^{\varphi_1}$ is the coset of the similarity U_1 defined by

$$xU_1 = b(\bar{a}x) = xL_{\bar{a}}L_b\,, \tag{7}$$

where L_c means left multiplication by c, while $\bar{U}^{\varphi_2}$ is the coset of the similarity U_2 defined by

$$xU_2 = (x\bar{a})b = xR_{\bar{a}}R_b \tag{8}$$

(see [2], Thm. 1).

Obviously $\bar{U}^{\varphi_1}$ and $\bar{U}^{\varphi_2}$ have order 2, since φ_1 and φ_2 are automorphisms, but they do not contain (6, 2) involutions because if ρ denotes the ratio of U^{φ_i}, we get $(U^{\varphi_i})^2 = -\rho$. These involutions of $PS^+(Q)$ were described in [5] as P-involutions. Now the ratio of $\bar{U}^{\varphi_1}$ is equal to the discriminant of the minus space of U and consequently if two cosets $\bar{U}_1^{\varphi_1}$ and $\bar{U}_2^{\varphi_1}$ have the same ratio they are conjugate in $PS^+(Q)$ since they are images of two conjugate (6,2) involutions. The same is true for $\bar{U}_1^{\varphi_2}$ and $\bar{U}_2^{\varphi_2}$. On the other hand,

$$xEL_aL_bE = \overline{(b(a\bar{x}))} = (x\bar{a})\bar{b} = xR_{\bar{a}}R_{\bar{b}} \tag{9}$$

says that the coset $\overline{L_aL_b}$ is conjugate to $\overline{R_{\bar{a}}R_{\bar{b}}}$ by $E \notin S^+(Q)$. Hence, the proof of ([5], lemma 4) implies that the cosets defined by similarities of the form (7) are distinct from the cosets defined by similarities of the form (8).

Let us define then $\{D_1\}$ as the subset of $PS^+(Q)$ consisting of cosets defined by similarities of the form (7) and $\{D_2\}$ as the subset of cosets of similarities of the form (8). Now we know that $\{D_0\}$, $\{D_1\}$, and $\{D_2\}$ are disjoint sets, and it follows from their definitions that

$$\{D_0\}^{\varphi_1} = \{D_1\}, \quad \{D_0\}^{\varphi_2} = \{D_2\}.$$

Moreover, $\varphi_1^2 = 1 = \varphi_2^2$ implies that

$$\{D_0\} = \{D_1\}^{\varphi_1}, \quad \{D_0\} = \{D_2\}^{\varphi_2}.$$

Using the relation $\varphi_2\varphi_1 = \varphi_1\epsilon$ we get

$$\{D_1\}^{\varphi_2} = \{D_1\}^{\varphi_1\epsilon\varphi_1} = \{D_0\}^{\epsilon\varphi_1} = \{D_0\}^{\varphi_1} = \{D_1\}$$

and similarly from $\varphi_1\varphi_2 = \varphi_2\epsilon$ we get $\{D_2\}^{\varphi_1} = \{D_2\}$.

So φ_1 can be described as the transposition (0, 1) and φ_2 as the transposition (0, 2). On the other hand, it follows from (9) that ϵ is the transposition (1, 2). Therefore any permutation of 0, 1, 2 can be described as a product of the element φ_1 and ϵ.

It is known that the only case in which the group $PS^+(Q)$ defined by a nondegenerate quadratic form over an 8-dimensional space has exceptional automorphisms is that in which Q is the quadratic form associated to a

Cayley algebra (see [5], Thm. 1). By an exceptional automorphism we mean one which is not of the form

$$\bar{U} \to \overline{S^{-1}US}$$

where $S \in \Gamma S(Q)$, the group of semisimilarities. The automorphisms of this kind form a group isomorphic to $P\Gamma S(Q)$ and they can be characterized as the automorphisms which take $\{D_0\}$ into itself (see [5], lemma 2). Moreover, these automorphisms together with any exceptional automorphism, such as φ_1, generate the whole group of automorphisms of $PS^+(Q)$ (see [5], Thm. 3). Since $P\Gamma S(Q)$ is generated by $P\Gamma S^+(Q)$ and $\bar{E}$, we know then that $P\Gamma S^+(Q)$, ϵ, and φ_1 generate the group of automorphisms of $PS^+(Q)$.

Now if we identify S_3 with its image under any isomorphism from S_3 to the group generated by φ_i and ϵ, since φ_1 and ϵ are automorphisms of $P\Gamma S^+(Q)$ we can define the semidirect product G of the group $P\Gamma S^+(Q)$ by S_3 by taking

$$(\sigma, \bar{S})(\sigma_1, \bar{S}_1) = (\sigma\sigma_1, \bar{S}^{\sigma_1}\bar{S}_1)$$

where $(\sigma, \bar{S}) \in S_3 \times P\Gamma S^+(Q)$.

THEOREM 2. *The group of automorphisms of $PS^+(Q)$ is isomorphic to the semidirect product of $P\Gamma S^+(Q)$ by S_3 defined above.*

Proof: It is clear that if we map $(\sigma, \bar{S})$ into the automorphism

$$\bar{U}^{(\sigma,\bar{S})} = \bar{S}^{-1}\bar{U}^{\sigma}\bar{S}$$

we get a homomorphism of G onto the group of automorphisms of $PS^+(Q)$ since the image of G includes all the automorphisms defined by $P\Gamma S(Q)$ and the exceptional automorphism φ_1. So we only have to prove that the mapping is an injection. Assume then that $(\sigma, \bar{S})$ is mapped into the identity and let $\bar{U}$ be a (6,2) coset of $PS^+(Q)$, then

$$\bar{U}^{(\sigma,\bar{S})} = \bar{S}^{-1}\bar{U}^{\sigma}\bar{S} = \bar{U} \quad .$$

implies that σ maps $\{D_0\}$ into itself. Hence σ is either the identity of S_3 or permutes $\{D_1\}$ and $\{D_2\}$ in which case $\sigma = \epsilon$. But if $\sigma = \epsilon$

$$\bar{T}^{(\epsilon,\bar{S})} = \bar{S}^{-1}\bar{E}\bar{T}\bar{E}\bar{S} = \bar{T}$$

for all $\bar{T}$ implies that $\bar{E}\bar{S}$ is the coset of the identity and this is impossible since E is an improper involution and S is proper. Hence $\sigma = 1$ and

$$\bar{T}^{(1,\bar{S})} = \bar{S}^{-1}\bar{T}\bar{S} = \bar{T}$$

implies that $\bar{S}$ is the coset of the identity, so that our mapping is $1 - 1$.

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